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Relativistic descriptions of few-body systems*

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Abstract A brief review of relativistic effects in few-body systems, of theoretical approaches, recent developments and applications is given. Manifestations of relativistic effects in the binding energies, in the electromagnetic form factors and in three-body observables are demonstrated. The three-body forces of relativistic origin are also discussed.

Keywords Few-body systems · Relativistic equations

1 Introduction

Light nuclei give a typical example of a few-body weakly bound system. Their binding energies B are of the order of 0.1% from their masses M. This however does not mean that the relativistic effects in light nuclei are also so small: they are much larger than the ratio B/M. The reason is that, in contrast to the hydrogen atom (for instance), the small nuclear binding energy is a results of cancellation of much more significant kinetic and potential energies. This can be illustrated as follows.

Let us consider a system of two non-relativistic particles interacting by the Yukawa potential: $V_{non-rel.}(r) = -\frac{\alpha}{r}\exp(-\mu r)$ that in the momentum space corresponds to the kernel:

$$V_{non-rel.}(\mathbf{q}) = \frac{-4\pi\alpha}{\mu^2+\mathbf{q}^2}. \tag{1}$$

This non-relativistic kernel is a limiting case of the following relativistic one-boson exchange kernel:

$$V_{rel.}(q) = \frac{-4\pi\alpha}{\mu^2-q^2-i\epsilon} = \frac{-4\pi\alpha}{\mu^2+\mathbf{q}^2-q_0^2-i\epsilon}. \tag{2}$$

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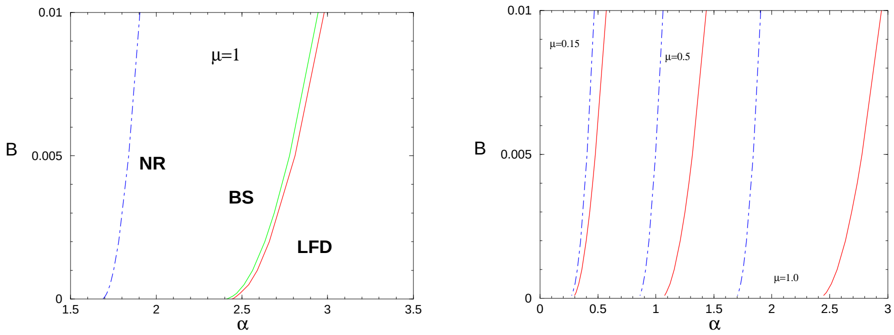

Fig. 1 Left: Binding energy B vs. coupling constant α, found via Shrödinger equation (NR), Bethe-Salpeter (BS) and the light-front (LFD) one, for exchange mass $\mu = 1$ in the kernels (1) and (2). Right: the same as at left, but for different exchange masses μ (here the BS and LFD results are indistinguishable from each other). The results are from [4].

$V_{rel.}$ turns into $V_{non-rel.}$ when $q_0 \to 0$. The kernel $V_{non-rel.}$ enters the Shrödinger equation. The kernel $V_{rel.}$ enters the relativistic Bethe-Salpeter (BS) equation [1]. Another popular relativistic approach is light-front dynamics (LFD). The corresponding equation – light front (LF) equation, see for review [2,3], – contains the kernel $V_{rel.}$, which analytical form differs from (2), but its non-relativistic limit also coincides with (1). All these three equations – Shrödinger equation, BS and LF ones – were solved, for spinless particles in the $J = 0$ state, in Ref. [4]. The results in the limit of extremely small binding energy $B \to 0$ are shown in fig. 1. Left panel corresponds to heavy exchange mass $\mu = 1$ (in the units of the constituent mass $m = 1$). We see that the relativistic calculations BS and LF are very close to each other and, at the same time, they strongly differ from the non-relativistic result NR. This shows that the relativistic effects can be important even at small binding energy. The curves at the right panel demonstrate that when the exchange mass decreases, the difference between relativistic and non-relativistic results decreases too. That is, the true relativistic system is a system not only with very small binding energy but also with interaction resulted from exchange by zero mass. When the exchange mass is not small (of the order of the constituent mass m), then the particles are in a very narrow potential well of the radius $r \sim 1/m$. Then their momenta are comparable with their masses $k \sim 1/r \sim m$, the kinetic energy (and the system at all) is relativistic and, hence, the small binding energy is a result of cancellation of large (positive) kinetic and large (negative) potential energies. That's why a system with small binding energy may be still relativistic. Similar situation is realized in nuclei, since the exchange mesons like ω and ρ are enough heavy ($\mu_\omega \approx \mu_\rho \approx 0.8\,m$).

In the opposite case of strongly bound systems, there is an example [5] that even a system with extremely large binding energy, such that its total mass tends to zero, may be dominated (by 90%) by a few lowest Fock sectors, containing two, three and four particles. It still remains to be a *few-body* system (though, highly relativistic).

Even in the case of mainly non-relativistic system (average momentum is very small), its impulse distribution contains a relativistic tail. This tail may be very small, but it completely determines the e.m. form factor at large momentum transfer. The

form factor of this system is also very small but it should be calculated in a relativistic approach.

All that requires development of appropriate approaches to relativistic description of few-body systems. Brief review of these approaches is given in the next section.

2 Relativistic descriptions

In non-relativistic quantum mechanics the wave function is an eigenvector of Hamiltonian: $H\psi = E\psi$. Dynamics is introduced by the adding to free Hamiltonian $H^{(0)}$ an interaction term $H^{int} = V$:

$$H^{(0)} \to H = H^{(0)} + H^{int} = \frac{p^2}{2m} + V(r).$$

In relativistic case, the relativistic covariance is guaranteed if the wave function (or the state vector $|p\rangle$) is forming a representation of the Poincaré group. The latter is determined by ten generators $P_\mu, J_{\mu\nu}$ which satisfy the following commutation relations:

$$\begin{aligned} &[P_\mu, P_\nu] = 0, \quad [P_\mu, J_{\kappa\rho}] = i(g_{\mu\rho}P_\kappa - g_{\mu\kappa}P_\rho), \\ &[J_{\mu\nu}, J_{\rho\gamma}] = i(g_{\mu\rho}J_{\nu\gamma} - g_{\nu\rho}J_{\mu\gamma} + g_{\nu\gamma}J_{\mu\rho} - g_{\mu\gamma}J_{\nu\rho}). \end{aligned} \tag{3}$$

The Hamiltonian $H = P_0$ is now only one of generators. Similarly to non-relativistic case, dynamics is introduced by the adding to free Poincaré generators an interaction terms in a way which keeps the commutation relations (3) unchanged. This is not simple but a solvable task. It can be realized in the framework of two different approaches: (*i*) relativistic quantum mechanics with fixed number of particles and (*ii*) field theory.

In relativistic quantum mechanics the Poincaré generators are the functions of fixed number (say, two or three) of the particles momenta. The interaction is a phenomenological one, it is fitted to describe e.g. the two-body phase shifts. Then one can make predictions: to calculate for instance the e.m. form factors or three-body observables. For good reviews of this approach see [6, 7].

In field theory the Poincaré generators are derived from Lagrangian by a well-known formulas, given almost in any textbook. If Lagrangian is not free (contains interaction), then the interaction appears also in the generators. The state vector $|p\rangle$, on which the generators act, can be decomposed in the basis of free fields (similarly to the Fourier decomposition in plane waves of the non-relativistic wave function). This basis is represented as an (infinite) set of Fock components with increasing numbers of particles. In practice, this decomposition is truncated (the desired number of particles is fixed by hand). After that the approach becomes approximate.

So, in practice, two approaches – (*i*) and (*ii*) – differ from each other by the point where one makes this truncation: (*i*) either from the very beginning, with further phenomenological construction the generators; (*ii*) or after finding the generators by the field theory recepees. In the latter case, the kernel is motivated by a field-theoretical Lagrangian. In its turn, this field-theoretical interaction is mainly restricted by the one-boson-exchange.

One should also distinguish three forms of relativistic dynamics, proposed by Dirac [8], which exist in both approaches. Namely: (a) instant form; (b) front form; (c) point form. They differ from each other by the ways of introducing the interaction in

generators. In the instant form the time component P_0 of the four-momentum operator P_μ contains interaction, whereas the spatial components P_j ($j = x, y, z$) are free. The interaction enters also in the components J_{0j} of the operators $J_{\mu\nu}$. The components J_{ij} are free. In the front form (LFD) the interaction enters the component $P_- = P_0 - P_z$, whereas the components $P_+ = P_0 + P_z,\ P_x,\ P_y$ are free. The operator $J_{\mu\nu}$ is constructed correspondingly. The components of this operator which transform the LF plane $t + z = const$ into itself are free. In the point form all the components of P_μ contain interaction, whereas the operator $J_{\mu\nu}$ is free.

In its turn, LFD is developed in two forms: ordinary LFD with the LF plane $t + z = const$ (see for review [3]) and explicitly covariant LFD [2] with the LF plane given by the invariant equation $\omega\cdot x = \omega_0 t - \boldsymbol{\omega}\cdot\mathbf{x} = const$, where $\omega = (\omega_0, \boldsymbol{\omega})$ is a four-vector with $\omega^2 = 0$. The main advantage of this latter formulation is in the fact that the dependence of the state vector on the LF orientation is given explicitly, in terms of the four-vector ω (see e.g. the LF deuteron wave function (7) below). In the particular case $\omega = (1, 0, 0, -1)$ we come back to the ordinary formulation of LFD.

The operator of e.m. current j_μ used to calculate e.m. form factors, like any four-vector operator, has the following commutation relation with $J_{\kappa\rho}$:

$$[j_\mu, J_{\kappa\rho}] = i(g_{\mu\rho} j_\kappa - g_{\mu\kappa} j_\rho). \tag{4}$$

If $J_{\kappa\rho}$ contains interaction (in l.h.-side of (4)), then r.h.-side of (4), i.e. j_μ, also must contain interaction. This means that in interacting system the exact e.m. current cannot be free (except for the point form of dynamics).

Another series of (field-theoretical) relativistic approached deals not with the state vector $|p\rangle$ itself, but it is based on the BS amplitude [1] defined as:

$$\Phi(x_1, x_2, p) = \langle 0|T\left(\varphi(x_1)\varphi(x_2)\right)|p\rangle, \tag{5}$$

$\varphi(x)$ is the Heisenberg field operator and $\langle 0|$ is the vacuum state. In the momentum space:

$$\Phi = \Phi(k_1, k_2, p) = \Phi(k, p), \quad p^2 = M^2,\ k_1^2 \neq m^2,\ k_2^2 \neq m^2 \quad \text{and} \quad k = (k_1 - k_2)/2.$$

The BS equation for Φ is singular, that complicates its numerical solution. To avoid singularities, one can transform this equation in the Euclidean space. Corresponding solution provides the binding energies. However, to calculate e.m. form factors, we should know the BS amplitude in Minkowski space. The Wick rotation in terms of the relative momentum k – the argument of the BS amplitude $\Phi(k, p)$ – is not valid in the form factor integral over k.

The methods to solve BS equation in Minkowski space were recently developed first for the spinless particles [9] and then for two fermions [10]. They are based on the Nakanishi integral representation [11]:

$$\Phi(k, p) = \int_{-1}^{1} \mathrm{d}z' \int_0^\infty \mathrm{d}\gamma' \frac{g(\gamma', z')}{\left[k^2 + p\cdot k\, z' + \frac{1}{4}M^2 - m^2 - \gamma' + i\epsilon\right]^3}. \tag{6}$$

This integral determines a singular BS amplitude $\Phi(k, p)$. However, the Nakanishi weight function $g(\gamma, z)$ is not singular. Substituting BS amplitude in the form (6) in the BS equation, one can derive and solve numerically equation for $g(\gamma, z)$. Then again using (6) with known $g(\gamma, z)$, one can express the observables, like e.m. form factors, through $g(\gamma, z)$ analytically and then compute them numerically. Another method to

solve the BS in Minkowski space is based on the separable approximation of the kernel (see [12] and references therein).

There are also a few reductions of the four-dimensional BS amplitude to a three-dimensional form (still in the Minkowski space). In this direction, the approach proposed in [13] (covariant spectator theory) is most advanced and well developed. In the covariant spectator theory, the NN potential was fitted and applied to the deuteron and three-body problems as well as to the e.m. form factors [14,15].

The theoretical activity in studying the relativistic few-body systems flourishes in all the forms of dynamics and in all the approaches listed above.

3 Some applications

It is clearly demonstrated (see e.g. [14]) that the non-relativistic calculations of the *ed* elastic cross section do not describe the data at $Q^2 \geq 1\ GeV^2/c^2$. One needs to perform the calculations with true relativistic deuteron wave function. In the non-relativistic case the latter is determined by two spin components: S- and D-waves. In relativistic approaches the number of components depends on approach. In the spectator theory [13], there are four components. There are six components in covariant LFD [16]. The deuteron BS amplitude is determined by eight components (see e.g. [2]).

As an example, we mention the calculation carried out in the framework of explicitly covariant LFD. In this approach, the relativistic deuteron wave function has the form [16]:

$$\boldsymbol{\psi}(\mathbf{k},\mathbf{n}) = f_1\frac{1}{\sqrt{2}}\boldsymbol{\sigma} + f_2\frac{1}{2}(\frac{3\mathbf{k}(\mathbf{k}\cdot\boldsymbol{\sigma})}{\mathbf{k}^2} - \boldsymbol{\sigma}) + f_3\frac{1}{2}(3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\sigma}) - \boldsymbol{\sigma}) \tag{7}$$

$$+f_4\frac{1}{2k}(3\mathbf{k}(\mathbf{n}\cdot\boldsymbol{\sigma}) + 3\mathbf{n}(\mathbf{k}\cdot\boldsymbol{\sigma}) - 2(\mathbf{k}\cdot\mathbf{n})\boldsymbol{\sigma}) + f_5\sqrt{\frac{3}{2}}\frac{i}{k}[\mathbf{k}\times\mathbf{n}] + f_6\frac{\sqrt{3}}{2k}[[\mathbf{k}\times\mathbf{n}]\times\boldsymbol{\sigma}],$$

where $\mathbf{n} = \boldsymbol{\omega}/|\boldsymbol{\omega}|$ and $\boldsymbol{\sigma}$ are the Pauli matrices. The vector $\mathbf{n}$ just provides the explicit dependence of this wave function on the LF orientation. The six components f_{1-6} were calculated in [17]. Only three of them dominate: f_1, f_2 (which turn into the S- and D-waves in non-relativistic limit) and f_5, whereas f_3, f_4, f_6 are negligible. The corresponding deuteron e.m. form factors were calculated in [18]. The results of this calculation are in good coincidence with the appeared later experimental data [19]. We do not give this comparison here. A detailed review can be found in [14]. One can conclude that the relativistic effects in a two-body system are taken into account satisfactory.

On the contrary, there are still the problems in the theoretical descriptions of the three-body systems. Though the problems with binding energy of tritium (underbinding) can be removed by incorporating the three-body forces, there are some deviations in description of the elastic *pd* scattering. They are seen in fig. 2 taken from [20,21].

There is also a discrepancy in the analyzing power A_y in the *pd* elastic scattering (see left panel in fig. 3 taken from [22]). Right panel [23] shows the *nd* breakup cross section: $nd \to (nn)p$ in a particular kinematics corresponding to so-called symmetric space-star configuration. In both cases, there are considerable deviations between different versions of the theoretical calculations and experimental data. One hopes to resolve these contradictions, properly taking into account relativistic effects as well as three-body forces. We will see below that the three-body forces can be partially

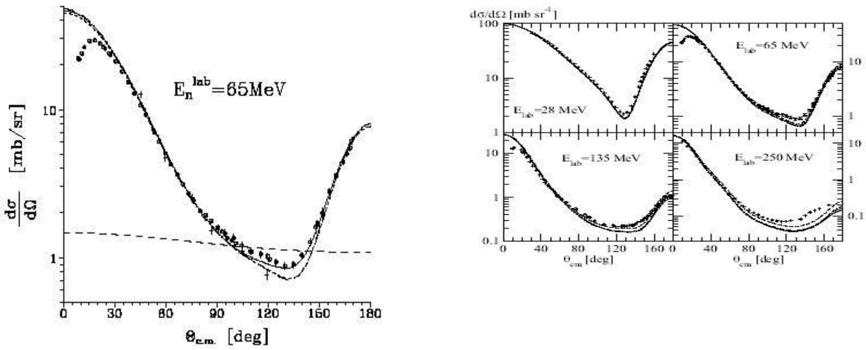

Fig. 2 Left: Cross section of elastic *nd* scattering vs. c.m. scattering angle. Short dashed line – non-relativistic calculation. Solid line – non-relativistic + 3-body forces. The figure is taken from [20]. Right: The same as at left for other energies. Solid line – non-relativistic calculation. Dashed line – relativistic one. The figure is taken from [21].

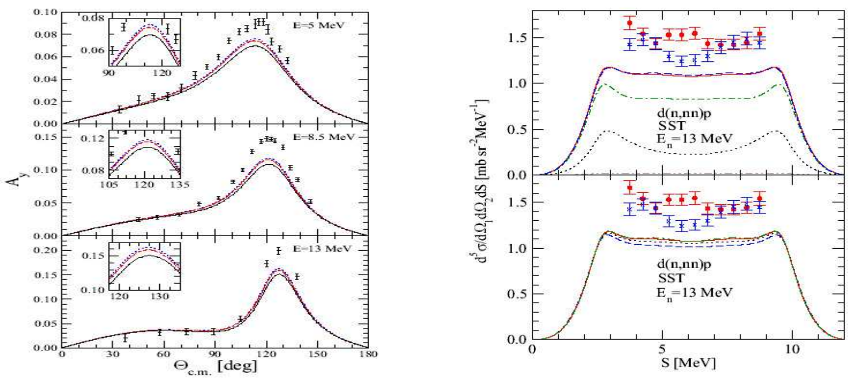

Fig. 3 Left: Analyzing power in elastic *pd* scattering vs. c.m. scattering angle. Dotted line – non-relativistic calculation. Solid line – relativistic one. The figure is taken from [22]. Right: The cross section of the *nd* breakup: $nd \to (nn)p$. The figure is taken from [23].

induced by relativity. The importance of relativistic effects in exclusive *pd* breakup scattering at intermediate energies was demonstrated in [24, 25], where the relativistic Faddeev equation was solved without employing a partial wave decomposition. The relativistic effects improve agreement with experimental data. The magnitude of these effects depends on configuration in the final state. Some success in describing A_y was achieved in [26].

4 Relativity in three-body systems

The binding energy of two-body system interacting by a potential described by the potential well U_0 with radius r_0 tends to constant when $U_0 \to \infty$, $r_0 \to 0$ but $U_0 r_0^2 = const$. On the contrary, for this interaction, the binding energy of three-body system

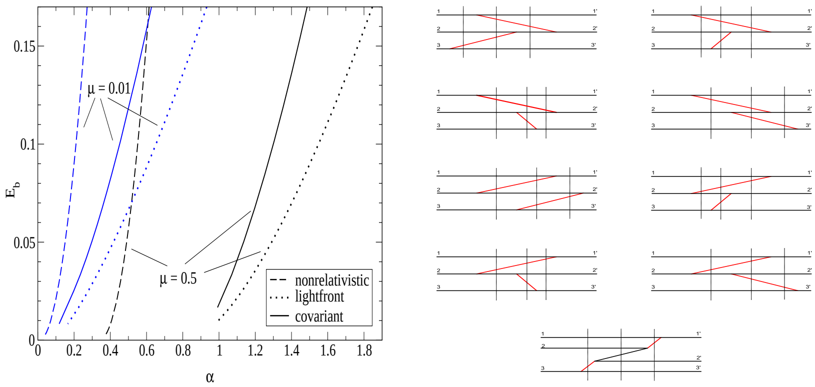

Fig. 4 Left: Three-body binding energy B vs. coupling constant α, found via Shrödinger equation (long-dashed), Bethe-Salpeter (solid) and the light-front (short-deshed), for exchange masses $\mu = 0.01$ and $\mu = 0.5$. The results are from [30]. Right: The graphs contributing in the three-body forces of relativistic origin.

tends to $-\infty$. This is a well-known property of non-relativistic three-body system which is called the Thomas collapse [27].

It turned out that the relativity results in an effective repulsion: for given two-body mass M_2, the three-body mass M_3 is finite [28,29]. This drastic change of three-body binding energy shows that the influence of relativity on three-body system may be stronger than on the two-body one.

However, for enough strong interaction, corresponding to large two-body binding energy, such that $M_2 < M_c = 1.43\, m$, the three-body mass becomes negative. In this domain of M_2, a physical solution for the three-body system disappears. In the non-relativistic scale, the binding energy equal to the total mass of constituents, is almost infinity. Therefore the case, when M_3, though being finite, approaches to zero, is a relativistic counterpart of Thomas collapse.

The relativistic three-body equations – BS and LF ones – have been also solved, for spinless particles, not only for zero range interaction, but also in more realistic case of one-boson exchange [30]. The corresponding two-body solution for binding energy [4] was discussed above and is shown in fig. 1. The three-body binding energy vs. coupling constant α is shown in fig. 4 (left panel). In contrast to the two-body results (see fig. 1), the BS and LF calculations do not coincide, but considerably differ from each other. They both also differ from the non-relativistic result (like in the two-body case). However, for the two-body system, when the exchange mass μ tends to zero, the BS and LF calculations (which are very close to each other) tend to the non-relativistic result. In three-body system this is not the case. The reason of these deviations is the three-body forces generated by relativity. Corresponding graphs, containing two mesons in flight (first considered in [31]), are shown in the right panel of fig. 4. They are automatically included in the three-body BS equation. However, they should be added explicitly in the kernel of the LF equation. After taking them into account [30], we find good coincidence between the BS and LF results (see fig. 5). This explicitly demonstrates that in a three-body system (a) relativistic effects and three-body forces appear together; (b) both may be important. Notice that the role of three-body forces may be different in different relativistic approaches. Thus, the relativistic three-body forces are not generated as a correction to the three-body spectator equation [13,15].

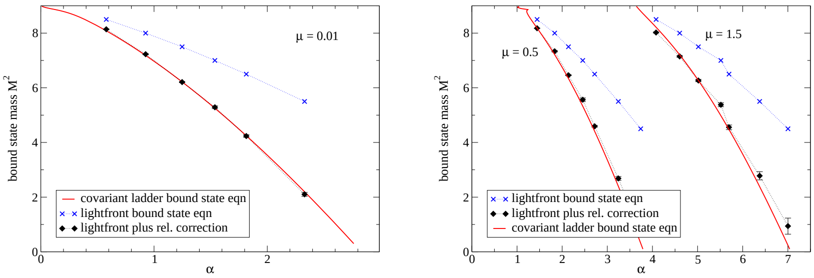

Fig. 5 Three-body bound state mass squared M_2^3 vs. coupling constant α for exchange masses $\mu = 0.01$ (left), $\mu = 0.5$ and $\mu = 1.5$ (right). The units are set by the constituent mass: $m = 1$. The figures are taken from [30].

However, they should be still incorporated as a relativistic correction to the Schrödinger equation.

The relativity is not the only source of the three-body forces. There exist other sources (e.g., the intermediate isobar creation) which may generate "intrinsic" three-body forces. One should include all that in the analysis of the discrepancies in three-body reactions discussed above.

5 Conclusions

We conclude that relativistic effects in nuclei can be important in spite of small binding energy. At high momenta they clearly manifest themselves and are necessary to describe the deuteron e.m. form factors. At the same time, there is still a discrepancy in three-body observables which might be a result of less clarity in understanding the corresponding relativistic effects, the relativistic NN kernel and the three-body forces.

Relativistic few-body physics remains to be a field of very intensive and fruitful researches.

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